

# ON THE ORBITAL STABILITY OF FRACTIONAL SCHRÖDINGER EQUATIONS

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**ABSTRACT.** We show the existence of ground state and orbital stability of standing waves of fractional Schrödinger equations with power type nonlinearity. For this purpose we establish the uniqueness of weak solutions.

## 1. INTRODUCTION

In this paper we consider the following Cauchy problem:

$$(1.1) \quad \begin{cases} i\partial_t \Phi + (-\Delta)^s \Phi = N(x, \Phi) & \text{in } \mathbb{R}^{1+n}, \\ \Phi(0, x) = \varphi(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Here  $n \geq 1$ ,  $0 < s < 1$ ,  $\Phi : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$  and  $N : \mathbb{R}^n \times \mathbb{C} \rightarrow \mathbb{C}$ .

The equation (1.1), called *fractional* nonlinear Schrödinger equation, appears in many fields in science and engineering. Other domains of applications of such equations, involving the fractional powers of the Laplacian, arise in medicine (RMI and heart diseases). It is also of a great importance in astrophysics, signal processing, turbulence, and water waves, where the cases  $s = \frac{1}{4}$  and  $s = \frac{3}{4}$  are the most relevant (see [12] and references therein).

We will focus our attention on the orbital stability of standing waves of this Schrödinger equation. Our results generalize those of [1] and [9]. The paper [1] seems to be the first one dealing with the orbital stability in the fractional case. The authors studied (1.1) for  $s = \frac{1}{2}, n = 1$  with an autonomous cubic power nonlinearity. In [9] the authors extended the previous paper to general nonlinearities for  $n \geq 2$ , but without showing the uniqueness of weak solutions. In our work, we use the concentration-compactness lemma to prove the orbital stability of standing waves, as stated by Cazenave and Lions, but without introducing a problem at infinity. See Proposition 1.1 below. We also establish the uniqueness of weak solutions to the Cauchy problem (1.1) under suitable conditions on  $V$  and  $f$ . Unlike the usual Schrödinger equation ( $s = 1$ ), it is not an easy matter to show the uniqueness in the fractional setting, since we cannot utilize the standard Strichartz estimates due to a regularity loss ([7]). Here we exploit weighted Strichartz estimates without regularity loss. Instead, some integrability conditions on  $a, V$  are necessary to treat the weights. The details of uniqueness of weak solutions will be discussed in Section 4.2.

To present our results let us set  $N(x, \Phi) = V(x)\Phi + f(x, \Phi)$  and describe assumptions: The functions  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f : \mathbb{R}^n \times \mathbb{C} \rightarrow \mathbb{C}$  are measurable and  $f$  satisfies that  $f(x, z) = \frac{z}{|z|}f(x, |z|)$  for  $z \in \mathbb{C} \setminus \{0\}$ , and for some  $\ell$  with  $0 < \ell \leq \frac{4s}{n-2s}$  and nonnegative

measurable functions  $a, b$

$$(1.2) \quad |f(x, z)| \leq a(x)|z|^{\ell+1},$$

for all  $x \in \mathbb{R}^n$  and

$$(1.3) \quad |f(x, z_1) - f(x, z_2)| \leq b(x)(|z_1|^\ell + |z_2|^\ell)|z_1 - z_2|.$$

for all  $x \in \mathbb{R}^n$  and  $z_1, z_2 \in \mathbb{C}$ . Let us also set  $F(x, |\Phi|) = \int_0^{|\Phi|} f(x, \alpha) d\alpha$ . Then we define a functional  $J$  by

$$J(\Phi) = \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2 - \frac{1}{2} \int V(x) |\Phi|^2 dx - \int F(x, |\Phi|) dx$$

and also  $M$  by  $M(\Phi) = \int |\Phi|^2 dx$ . By a standing wave of (1.1) we mean a solution  $\Phi(t, x)$  of the form  $e^{i\omega t} u$  for some  $\omega \in \mathbb{R}$ , where  $u$  is a solution of the equation

$$(1.4) \quad (-\Delta)^s u - \omega u = V(x)u + f(x, u).$$

Some authors have studied the existence of  $u$  under suitable conditions on  $f$ . For this purpose they showed that if  $(u_k)$  is a minimizing sequence of the problem

$$I_\mu = \inf\{J(u) : u \in S_\mu\}, \quad S_\mu = \{u \in H^s(\mathbb{R}^n, \mathbb{C}) : M(u) = \mu\}$$

with a prescribed positive number  $\mu$ , then  $u_k \rightarrow u$  in  $H^s$  up to a subsequence, where  $u$  is a solution of (1.4) for some  $\omega$ . Now by following the definition of Cazenave-Lions, we set

$$\mathcal{O}_\mu = \{u \in S_\mu : J(u) = I_\mu\}.$$

Our first result is the existence of ground states.

**Proposition 1.1.** *Let  $n \geq 1$ ,  $0 < s < 1$  and  $0 < \ell < \frac{4s}{n}$ . Suppose that*

$$0 \leq V \in L_{loc}^{p_1} + L^{p_2}(|x| > 1)$$

*for  $\frac{n}{2s} < p_1, p_2 < \infty$  and  $f$  satisfies (1.2) with  $a \in L_{loc}^{q_1} + L^{q_2}(|x| > 1)$  for  $\frac{2n}{4s-n\ell} < q_1, q_2 < \infty$ , and that there exist  $\kappa, R, N, \delta > 0$  and  $\beta, \sigma > 0$  such that*

$$\frac{n\beta}{2} + \delta - 2s < 0$$

*and*

$$(1.5) \quad F(x, |z|) \geq \kappa|x|^{-\delta}|z|^{2+\beta},$$

*for any  $|z| \leq N$  and  $|x| \geq R$  and*

$$(1.6) \quad F(x, \theta|z|) \geq \theta^{2+\sigma} F(x, |z|)$$

*for all  $x, z, \theta > 1$ . Then  $\mathcal{O}_\mu$  is not empty for any  $\mu > 0$ . When  $\ell = \frac{4s}{n}$ , we assume that  $a \in L^q(|x| \geq 1) \cap L^\infty$  for some  $q$  with  $\frac{n^2}{4s^2} < q < \infty$  if  $n \geq 2$  and  $1 < q < \infty$  if  $n = 1$ . Then  $\mathcal{O}_\mu$  is not empty for sufficiently small  $\mu > 0$ .*

When  $a \in L^\infty$ , by adding some natural asymptotic conditions on  $V$  and  $f$ , one can show the existence of ground state. See [8] and [11]. But if we assume the radial symmetry of  $V(x), f(x, \cdot)$  in  $x$ , then using the compactness of embedding  $H_{rad}^s \hookrightarrow L^p$ ,  $\frac{1}{2} < s < \frac{n}{2}$ ,  $2 < p < \frac{2n}{n-2s}$  (see [6]), we have the following.

**Proposition 1.2.** *Let  $n \geq 2$ ,  $\frac{1}{2} < s < 1$  and  $0 < \ell < \frac{4s}{n}$ . Let  $0 \leq V \in L_{rad,loc}^\infty$  satisfy*

$$(1.7) \quad \int_{|x|>1} V(x)|x|^{-(n-2s)} dx < \infty$$

*and  $f(x, \cdot)$  be radially symmetric with  $a \in L^\infty$ . If  $F$  satisfies (1.5) and (1.6), then  $\mathcal{O}_\mu \cap H_{rad}^s$  is not empty for any  $\mu > 0$ . When  $\ell = \frac{4s}{n}$ ,  $\mathcal{O}_\mu$  is not empty for sufficiently small  $\mu > 0$ .*

We say that  $\mathcal{O}_\mu$  is stable if it is not empty and satisfies that for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $\varphi \in H^s$  with

$$\inf_{u \in \mathcal{O}_\mu} \|\varphi - u\|_{H^s} < \delta,$$

then

$$\inf_{u \in \mathcal{O}_\mu} \|\Phi(t, \cdot) - u\|_{H^s} < \varepsilon$$

for all  $t \in [-T_1, T_2]$ . Here  $\Phi$  is the unique solution to (1.1) in  $C([-T_1, T_2]; H^s)$  with  $M(\Phi(t)) = M(\varphi)$  and  $J(\Phi(t)) = J(\varphi)$  for all  $t \in [-T_1, T_2]$ .

Let us introduce our main result.

**Theorem 1.3.** *Suppose that  $s, \ell, V$  and  $f$  satisfy all conditions as in Proposition 1.1, and that (1.1) has the unique solution in  $C([-T_1, T_2]; H^s)$  with  $M(\Phi(t)) = M(\varphi)$  and  $J(\Phi(t)) = J(\varphi)$  for all  $t \in [-T_1, T_2]$ . Then  $\mathcal{O}_\mu$  is stable.*

**Theorem 1.4.** *Suppose that  $s, \ell, V$  and  $f$  satisfy all conditions as in Proposition 1.2, and that (1.1) has the unique solution in  $C([-T_1, T_2]; H^s)$  with  $M(\Phi(t)) = M(\varphi)$  and  $J(\Phi(t)) = J(\varphi)$  for all  $t \in [-T_1, T_2]$ . Then  $\mathcal{O}_\mu \cap H_{rad}^s$  is stable.*

In view of the well-posedness results in Section 4.2 below, by assuming that  $V, a, b$  are smooth and have suitable decay at infinity, we get the orbital stability for  $\frac{1}{2} < s < 1$ ,  $0 < \ell \leq \frac{4s}{n}$  and  $\ell < \ell_0$ , where  $\ell_0 = \infty$ , if  $n = 1$ ,  $\frac{2s-1}{2s(1-s)}$  if  $n = 2$  and  $\frac{n(2s-1)}{(n-2s)(n-1)}$  if  $n \geq 3$ . The critical case  $\ell = \frac{4s}{n}$  can be included when  $n = 1, 2, 3$ .

Our paper is organized as follows. In Section 2 we will prove the existence of ground states by showing the compactness of the minimizing sequences of the constrained variational problem. This is a key step to show the orbital stability of standing waves. This goal is achieved in Theorem 1.3 and Theorem 1.4, which will be shown in Section 3. In the last section, we will discuss the uniqueness of solutions of the Cauchy problem for a large class of nonlinearities.

## 2. GROUND STATE

**2.1. Proof of Proposition 1.1.** If  $0 < \ell < \frac{4s}{n}$ ,  $p_1, p_2 > \frac{n}{2s}$  and  $q_1, q_2 > \frac{2n}{4s-n\ell}$ , then from Gagliardo-Nirenberg's and Young's inequalities it follows that for any  $u \in S_\mu$  there exist  $\lambda$  such that

$$\begin{aligned}
 J(u) &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \int V(x) |u|^2 dx - \int F(x, |u|) dx \\
 &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \|V\|_{L^{p_1}(|x| \leq 1)} \|u\|_{L^{2p'_1}}^2 - \|V\|_{L^{p_2}(|x| > 1)} \|u\|_{L^{2p'_2}}^2 \\
 &\quad - \|a\|_{L^{q_1}(|x| \leq 1)} \|u\|_{L^{(\ell+2)q'_1}}^{\ell+2} - \|a\|_{L^{q_2}(|x| > 1)} \|u\|_{L^{(\ell+2)q'_2}}^{\ell+2} \\
 &\geq \frac{1}{4} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \lambda \sum_i \mu^{\theta_i} \quad (\theta_1 = \theta_2 = 2, \theta_3 = \theta_4 = \ell).
 \end{aligned} \tag{2.1}$$

Thus  $I_\mu > -\infty$  for all  $\mu > 0$ . If  $0 < \ell < \frac{4s}{n}$ ,  $p_1, p_2 > \frac{n}{2s}$ ,  $a \in L^\infty$  then there exist  $\lambda, \theta_i > 0$ ,  $i = 1, 2$

$$\begin{aligned}
 J(u) &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \int V(x) |u|^2 dx - \int F(x, |u|) dx \\
 &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \|V\|_{L^{p_1}(|x| \leq 1)} \|u\|_{L^{2p'_1}}^2 - \|V\|_{L^{p_2}(|x| > 1)} \|u\|_{L^{2p'_2}}^2 \\
 &\quad - \|a\|_{L^\infty} \|u\|_{L^{(\ell+2)}}^{\ell+2} \\
 &\geq \frac{1}{4} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \lambda \sum_{i=1,2} \mu^{\theta_i} - C \|a\|_{L^\infty} \mu^{\frac{4s}{n}} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2.
 \end{aligned} \tag{2.2}$$

So, if  $C \|a\|_{L^\infty} \mu^{\frac{4s}{n}} < \frac{1}{4}$ , then  $I_\mu > -\infty$ .

We show that

$$I_\mu < 0 \quad \text{for all } \mu > 0. \tag{2.3}$$

In fact, for  $0 < \lambda \ll 1$  letting  $\psi_\lambda(x) = \lambda^{\frac{n}{2}} \psi(\lambda x)$  for a nonnegative, rapidly decreasing radial smooth function  $\psi$  in  $S_\mu$  with  $\psi \leq N$ , we see that  $\psi_\lambda \in S_\mu$  and

$$\begin{aligned}
 J(\psi_\lambda) &= \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} \psi_\lambda\|_{L^2}^2 - \frac{1}{2} \int V(x) (\psi_\lambda)^2 dx - \int F(x, \psi_\lambda) dx \\
 &\leq \lambda^{2s} \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} \psi\|_{L^2}^2 - \kappa \lambda^{\frac{n(\beta+2)}{2} - n + \delta} \int_{|x| \geq R} |x|^{-\delta} (\psi(x))^\beta dx.
 \end{aligned}$$

Since  $0 < \lambda \ll 1$  and  $\psi$  is smooth and rapidly decreasing, there exist constants  $C_1, C_2 > 0$  such that

$$J(\psi_\lambda) \leq \lambda^{2s} (C_1 - \lambda^{\frac{n\beta}{2} + \delta - 2s} C_2),$$

which is strictly negative from (1.5) if  $\lambda$  is sufficiently small.

On the other hand, from the proof of Lemma 3.1 of [11] one can easily show that  $I_\mu$  is continuous on  $(0, \infty)$ .

For each  $\mu > 0$  and  $\theta > 1$  we take  $\varepsilon < -I_\mu(1 - \theta^{-\frac{\sigma}{2}})$  and  $v \in S_\mu$  such that  $I_\mu < J(v) < I_\mu + \varepsilon$ . Then from (1.6) it follows that

$$I_{\theta\mu} \leq J(\sqrt{\theta}v) \leq \theta^{1+\frac{\sigma}{2}} J(v) \leq \theta^{1+\frac{\sigma}{2}} (I_\mu + \varepsilon) < \theta I_\mu,$$

which means

$$(2.4) \quad I_\mu < I_\nu + I_{\mu-\nu} \text{ for all } 0 < \nu < \mu.$$

For this see [14].

Let  $(u_j) \subset S_\mu$  be a minimizing sequence such that  $J(u_j) \rightarrow I_\mu$ . From (2.1) we deduce that  $(u_j)$  is bounded in  $H^s$ . To show  $\mathcal{O}_\mu \neq \emptyset$  we will use the concentration-compactness (see [14]). Let the concentration function  $\mathfrak{m}_j$  be defined by

$$\mathfrak{m}_j(r) = \sup_{y \in \mathbb{R}^n} \int_{|x-y| < r} |u_j(x)|^2 dx \text{ for } r > 0.$$

Set

$$\nu = \lim_{r \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathfrak{m}_j(r).$$

Then  $0 \leq \nu \leq \mu$  and there exists a subsequence  $u_j$  (still denoted by  $u_j$ ) satisfying the following properties<sup>1</sup>.

- (1) If  $\nu = 0$ , then  $\|u_j\|_{L^p} \rightarrow 0$  as  $j \rightarrow \infty$  for all  $p$  with  $2 < p < s^*$ ,  $s^* = \frac{2n}{n-2s}$  if  $n > 2s$  and  $s^* = \infty$  if  $n = 1$  and  $\frac{1}{2} \leq s < 1$ .
- (2) If  $\nu = \mu$ , then there exists a sequence  $(y_j) \subset \mathbb{R}^n$  and  $u \in H^s$  such that for any  $p$  with  $2 \leq p < s^*$

$$u_j(\cdot + y_j) \rightarrow u \text{ as } j \rightarrow \infty \text{ in } L^p$$

and given  $\varepsilon > 0$  there exists  $j_0(\varepsilon)$  and  $r(\varepsilon)$  such that

$$\int_{|x-y_j| < r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon, \text{ whenever } j \geq j_0(\varepsilon).$$

- (3) If  $0 < \nu < \mu$ , then there exist  $(v_j), (w_j) \subset H^s$  such that

$$(2.5) \quad \text{supp } v_j \cap \text{supp } w_j = \emptyset,$$

$$(2.6) \quad \|v_j\|_{H^s} + \|w_j\|_{H^s} \leq C \|u_j\|_{H^s},$$

$$(2.7) \quad \lim_{j \rightarrow \infty} M(v_j) = \nu, \quad \lim_{j \rightarrow \infty} M(w_j) = \mu - \nu,$$

$$(2.8) \quad \liminf_{j \rightarrow \infty} \left( \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2}^2 - \|(\Delta)^{\frac{s}{2}} v_j\|_{L^2}^2 - \|(\Delta)^{\frac{s}{2}} w_j\|_{L^2}^2 \right) \geq 0,$$

$$(2.9) \quad \lim_{j \rightarrow \infty} \|u_j - v_j - w_j\|_{L^p} = 0, \quad 2 \leq p < s^*.$$

If  $\nu = 0$ , then for  $0 < \ell < \frac{4s}{n}$  we have

$$\begin{aligned} & \frac{1}{2} \int V(x) |u_j|^2 dx + \int F(x, |u_j|) dx \\ & \leq C \|V\|_{L^{p_1} + L^{p_2}} \|u_j\|_{L^{2p'_1} + L^{2p'_2}}^2 + C \|a\|_{L^{q_1} + L^{q_2}} \|u_j\|_{L^{(\ell+2)q'_1} + L^{(\ell+2)q'_1}} \rightarrow 0 \end{aligned}$$

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<sup>1</sup>One can verify the concentration-compactness by following the arguments in [14] or [2]. We omit the details.

as  $j \rightarrow \infty$ . Here  $L^{l_1} + L^{l_2}$  denotes  $L^{l_1}(|x| \leq 1) + L^{l_2}(|x| > 1)$ . For  $\ell = \frac{4s}{n}$  we also have

$$\begin{aligned} & \frac{1}{2} \int V(x) |u_j|^2 dx + \int F(x, |u_j|) dx \\ & \leq C \|V\|_{L^{p_1} + L^{p_2}} \|u_j\|_{L^{2p'_1} + L^{2p'_2}}^2 + C \|a\|_{L^\infty} \|u_j\|_{L^{\frac{2n+4s}{n}}} \rightarrow 0 \end{aligned}$$

as  $j \rightarrow \infty$ . This implies  $I_\mu = \lim_{j \rightarrow \infty} J(u_j) \geq \frac{1}{2} \liminf \|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2}^2 \geq 0$  and contradicts (2.3).

If  $0 < \nu < \mu$ , then from the support condition (2.5) it follows that

$$\begin{aligned} & J(u_j) - J(v_j) - J(w_j) \\ & = \frac{1}{2} (\|(-\Delta)^{\frac{s}{2}} u_j\|_{L^2}^2 - \|(\Delta)^{\frac{s}{2}} v_j\|_{L^2}^2 - \|(\Delta)^{\frac{s}{2}} w_j\|_{L^2}^2) \\ & \quad - \frac{1}{2} \int V(x) (|u_j|^2 - |v_j + w_j|^2) dx - \int (F(x, |u_j|) - F(x, |v_j + w_j|)) dx. \end{aligned}$$

From (2.8) and (2.9) we deduce that

$$\liminf_{j \rightarrow \infty} (J(u_j) - J(v_j) - J(w_j)) \geq 0$$

and thus

$$I_\mu = \lim_{j \rightarrow \infty} J(u_j) \geq \liminf_{j \rightarrow \infty} J(v_j) + \liminf_{j \rightarrow \infty} J(w_j).$$

Since  $M(v_j) \rightarrow \nu$  and  $M(w_j) \rightarrow \mu - \nu$ , by the continuity of  $I_\mu$  on  $(0, \infty)$  we get

$$I_\mu \geq I_\nu + I_{\mu-\nu},$$

which contradicts (2.4).

Therefore  $\nu = \mu$ . Set  $\tilde{u}_j(x) = u_j(x + y_j)$ . Then  $u, \tilde{u}_j \in S_\mu$  and  $\tilde{u}_j \rightarrow u$  in  $L^p$  for all  $2 \leq p < s^*$ . On the other hand,  $(u_j)$  is bounded in  $H^s$ . So, there is a subsequence (still denoted by  $u_j$ ) converging to  $v$  weakly in  $H^s$  and strongly in  $L_{loc}^p$  for any  $1 \leq p < s^*$ . Now for any  $\varepsilon > 0$  we can find  $R, j_0 > 1$  such that

$$\begin{aligned} & \int_{|x| > R} V(x) (|u_j|^2 + |v|^2) dx \leq C \|V\|_{L^{p_2}(|x| > R)} < \frac{\varepsilon}{4}, \\ & \int_{|x| \leq R} V(x) (|u_j| + |v|) |u_j - v| dx \leq C \|V\|_{L^{p_1}(|x| \leq R)} \|u_j - v\|_{L^{2p'_1}(|x| \leq R)} < \frac{\varepsilon}{4}, \end{aligned}$$

(when  $0 < \ell < \frac{4s}{n}$ )

$$\begin{aligned} & \int_{|x| > R} a(x) (|u_j|^{\ell+2} + |v|^{\ell+2}) dx \leq C \|a\|_{L^{q_2}(|x| > R)} < \frac{\varepsilon}{4}, \\ & \int_{|x| \leq R} a(x) (|u_j|^{\ell+1} + |v|^{\ell+1}) |u_j - v| dx \leq C \|a\|_{L^{q_1}(|x| \leq R)} \|u_j - v\|_{L^{(\ell+2)q'_1}} < \frac{\varepsilon}{4}, \end{aligned}$$

(when  $\ell = \frac{4s}{n}$ )

$$\begin{aligned} & \int_{|x| > R} a(x) (|u_j|^{\ell+2} + |v|^{\ell+2}) dx \leq C \|a\|_{L^q(|x| > R)} < \frac{\varepsilon}{4}, \\ & \int_{|x| \leq R} a(x) (|u_j|^{\ell+1} + |v|^{\ell+1}) |u_j - v| dx \leq C \|a\|_{L^\infty(|x| \leq R)} \|u_j - v\|_{L^{\frac{2n+4s}{n}}} < \frac{\varepsilon}{4}, \end{aligned}$$

if  $j > j_0$ . Set  $P(w) \equiv J(w) - \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}w\|_{L^2}^2$ . Then  $P(u_j) \rightarrow P(v)$  as  $j \rightarrow \infty$ . Suppose that  $(y_j)$  is unbounded. Then up to subsequence we may assume that  $|y_j| \rightarrow \infty$ . Since  $\tilde{u}_j \rightarrow u$  in  $L^2$ ,  $u_j - u(\cdot - y_j) \rightarrow 0$  in the sense of distributions. But  $u(\cdot - y_j) \rightarrow 0$  and  $u_j \rightarrow v$  in the sense of distributions and thus  $v = 0$ . That is,  $P(u_j) \rightarrow 0$  as  $j \rightarrow \infty$ . This implies that  $I_\mu = \lim_{j \rightarrow \infty} J(u_j) \geq 0$ , which contradicts (2.3). So,  $(y_j)$  is bounded. Now let  $R^* = \sup_{j \geq 1} |y_j|$ . Then for any  $\varepsilon > 0$  we have

$$\int_{|x| < R^* + r(\varepsilon)} |u_j|^2 dx \geq \int_{|x - y_j| < r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon, \quad \text{if } j \geq j_0.$$

and thus

$$M(v) \geq \int_{|x| < R^* + r(\varepsilon)} |v|^2 dx \geq \lim_{j \rightarrow \infty} \int_{|x| < R^* + r(\varepsilon)} |u_j|^2 dx \geq \mu - \varepsilon.$$

This means  $M(v) \geq \mu$ , while the semi-continuity of weak limit implies  $M(v) \leq \mu$ . Then  $v \in S_\mu$ . Since  $P(u_j) \rightarrow P(v)$ , we have

$$(2.10) \quad I_\mu \leq J(v) \leq \liminf \|(-\Delta)^{\frac{s}{2}}u_j\|_{L^2}^2 + P(v) = \liminf (J(u_j)) = I_\mu.$$

Therefore  $J(v) = I_\mu$ . This completes the proof of Proposition 1.1.

**2.2. Proof of Proposition 1.2.** Set  $\overline{S}_\mu = \{v \in H_{rad}^s : M(v) = \mu\}$  and  $\overline{I}_\mu = \inf_{v \in \overline{S}_\mu} J(v)$ .

From Proposition 1 of [6] we see that

$$|u(x)| \leq C|x|^{-(\frac{n}{2}-s)} \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}$$

a.e  $x \in \mathbb{R}^n$ , if  $u \in H_{rad}^s$ ,  $\frac{1}{2} < s < \frac{n}{2}$ . So, given  $\varepsilon > 0$  we can find an  $R > 0$  such that  $\int_{|x| > R} V(x)|x|^{-(n-2s)} dx < \varepsilon$ . From this we get

$$\int_{|x| > R} V(x)|u|^2 dx \leq C \int_{|x| > R} V(x)|x|^{-(n-2s)} dx \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2 \leq C\varepsilon \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2.$$

In view of (2.1) and (2.2),  $\overline{I}_\mu > -\infty$ . From (1.5) and (1.6) it follows that  $\overline{I}_\mu < 0$  and  $I_\mu < I_\nu + I_{\mu-\nu}$  for all  $0 < \nu < \mu$ . Now let us take a minimizing sequence  $(u_j) \subset \overline{S}_\mu$  such that  $J(u_j) \rightarrow \overline{I}_\mu$ . Then from the concentration-compactness we get (1), (2), (3) with  $y_j = 0$ . By the same argument as above it follows that there exists a function  $u \in H^s$  such that  $J(u) = \overline{I}_\mu$ , provided we can show that there exists a subsequence  $(u_j)$  (denoted by  $u_j$  again)  $P(u_j) \rightarrow P(u)$  as  $j \rightarrow \infty$ . In fact, by the compact embedding  $H^s \hookrightarrow L^p$ ,  $2 < p < \frac{2n}{n-2s}$ , we can find a subsequence  $(u_j)$  such that  $u_j \rightarrow u$  in  $L^p$ . So, it is clear that  $\int F(x, |u_j|) dx \rightarrow \int F(x, |u|) dx$  for  $a \in L^\infty$ . Now it remains to show that  $\int V|u_j|^2 dx \rightarrow \int V|u|^2 dx$ . By (1.7) we deduce that for any  $\varepsilon > 0$  there is an  $R > 0$  such that

$$\int_{|x| > R} V|u_j|^2 dx \leq C \int_{|x| > R} V(x)|x|^{-(n-2s)} dx \|(-\Delta)^{\frac{s}{2}}u_j\|_{L^2}^2 \leq \frac{\varepsilon}{2}.$$

Since  $V \in L_{loc}^\infty$  and the embedding  $H^s \hookrightarrow L_{loc}^2$  is compact, (up to a subsequence) there exists  $j_0 > 0$  such that

$$\left| \int_{|x| \leq R} V|u_j|^2 dx - \int_{|x| \leq R} V|u|^2 dx \right| \leq 2\sqrt{c} \|V\|_{L^\infty(|x| \leq R)} \|u_j - u\|_{L^2(|x| \leq R)} \leq \frac{\varepsilon}{2}$$

if  $j \geq j_0$ . This completes the proof of Proposition 1.2.

### 3. PROOF OF THEOREMS 1.3, 1.4

Since the proofs of Theorems 1.3, 1.4 are quite the same, we only consider the proof of Theorem 1.3. The proof proceeds by contradiction. Suppose that  $\mathcal{O}_\mu$  is not stable, then either  $\mathcal{O}_\mu$  is empty or there exist  $w \in \mathcal{O}_\mu$  and a sequence  $\Phi_0^j \in H^s$  such that

$$\|\Phi_0^j - w\|_{H^s} \rightarrow 0 \text{ as } j \rightarrow \infty$$

but

$$(3.1) \quad \inf_{v \in \mathcal{O}_\mu} \|\Phi^j(t_j, \cdot) - v\|_{H^s} \geq \varepsilon_0$$

for some sequence  $t_j \in \mathbb{R}$  and  $\varepsilon_0$ , where  $\Phi^j(t, \cdot)$  is the solution of (1.1) corresponding to the initial data  $\Phi_0^j$ . Let  $w_j = \Phi^j(t_j, \cdot)$ . Since  $w \in S_\mu$  and  $J(w) = I_\mu$ , it follows from the continuity of  $L^2$  norm and  $J$  in  $H^s$  that

$$\|\Phi_0^j\|_{L^2} \rightarrow \mu \text{ and } J(\Phi_0^j) \rightarrow I_\mu.$$

Thus we deduce from the conservation laws that

$$\|w_j\|_{L^2} = \|\Phi_0^j\|_{L^2} \rightarrow \mu, \quad J(w_j) = J(\Phi_0^j) \rightarrow I_\mu.$$

Therefore if  $w_j$  has a subsequence converging to an element  $w \in H^s$  such that  $\|w\|_{L^2} = \mu$  and  $J(w) = I_\mu$ . This shows that  $w \in \mathcal{O}_\mu$  but

$$\inf_{v \in \mathcal{O}_\mu} \|\Phi^j(t_j, \cdot) - v\|_{H^s} \leq \|w_j - w\|_{H^s},$$

which contradicts (3.1). Since  $\mathcal{O}_\mu$  is not empty, to show the orbital stability of  $\mathcal{O}_\mu$  one has to prove that any sequence  $(w_j) \subset H^s$  with

$$(3.2) \quad \|w_j\|_{L^2} \rightarrow \mu \text{ and } J(w_j) \rightarrow I_\mu$$

is relatively compact in  $H^s$ . Since  $I_\mu$  is continuous w.r.t  $\mu \in (0, \infty)$  and  $\ell \leq \frac{4s}{n}$ , by the arguments in the proof of Proposition 1.1 we may assume that  $(w_j)$  is bounded in  $H^s$  and also verify from (2.10) that by passing to a subsequence there exists  $w \in H^s$  such that

$$(3.3) \quad w_j \rightharpoonup w \text{ in } H^s \text{ and } \lim_{j \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}} w_j\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} w\|_{L^2}.$$

This implies  $w_j \rightarrow w$  in  $H^s$  and thus the relative compactness.

### 4. UNIQUENESS AND WELL-POSEDNESS

In this section we show the existence of weak solutions and its uniqueness.



**4.1. Uniqueness of weak solution.** We first consider the existence of weak solutions to (1.1).

**Proposition 4.1.** *Let  $n \geq 1$ ,  $0 < s < 1$  and  $0 < \ell < s^* - 2$ ,  $s^* = \frac{2n}{n-2s}$  if  $n > 2s$  and  $s^* = \infty$  if  $n = 1$  and  $\frac{1}{2} \leq s < 1$ . Let  $V \in L_{loc}^{p_1} + L^{p_2}(|x| > 1)$  for  $\frac{n}{2s} < p_1, p_2 \leq \infty$ . Let  $f$  satisfy (1.2) and (1.3) with  $a, b \in L_{loc}^{q_1} + L^{q_2}(|x| > 1)$  for  $q_0 < q_1, q_2 \leq \infty$ , where  $q_0 = \frac{2n}{2n-(\ell+2)(n-2s)}$  if  $n > 2s$ , and  $q_0 = 1$  if  $n = 1$  and  $\frac{1}{2} \leq s < 1$ . Then there exists a weak solution  $\Phi$  such that*

$$\Phi \in L^\infty(-T_{min}, T_{max}; H^s) \cap W^{1,\infty}(-T_{min}, T_{max}; H^{-s}),$$

$$M(\Phi(t)) = M(\varphi), \quad J(\Phi(t)) \leq J(\varphi)$$

for all  $t \in (-T_{min}, T_{max})$ , where  $(-T_{min}, T_{max})$  is the maximal existence time interval of  $\Phi$  for given initial data  $\varphi$ .

*Proof of Proposition 4.1.* To show the existence of weak solutions we follow the standard regularizing argument (for instance see [2]). For this purpose we have only to verify that  $\|N(x, \Phi) - N(x, \Psi)\|_{H^{-s}} \leq C(K)\|\Phi - \Psi\|_{H^s}$ , provided  $\|\Phi\|_{H^s} + \|\Psi\|_{H^s} \leq K$ . In fact, since  $p_i > \frac{n}{2s}$ ,  $i = 1, 2$ , we can always find  $r_i, l_i \in [2, s^*)$  such that

$$\|V(\Phi - \Psi)\|_{L^{l'_1} + L^{l'_2}} \leq \|V\|_{L^{p_1} + L^{p_2}} \|\Phi - \Psi\|_{L^{r_1} + L^{r_2}}$$

for  $\frac{1}{r_i} = 1 - \frac{1}{l_i} - \frac{1}{p_i}$ . If  $n > 2s$  and  $q_0 < q_i \leq \infty$ ,

$$\begin{aligned} & \|f(x, \Phi) - f(x, \Psi)\|_{L^{\tilde{l}_1} + L^{\tilde{l}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (\|\Phi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}}^\ell + \|\Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}}^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (2K^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}} \end{aligned}$$

for  $\frac{\ell+1}{\tilde{r}_i} = 1 - \frac{1}{q_i} - \frac{1}{l_i}$ . Here we used the Sobolev embedding  $H^s \hookrightarrow L^{\tilde{r}_i}$ . If  $n = 1$  and  $\frac{1}{2} < s < 1$ , for any  $1 < q_1, q_2 \leq \infty$  we can find  $\tilde{r}_i, \tilde{l}_i \in [2, \infty)$  such that  $\frac{1}{\tilde{l}_i} = \frac{1}{q_i} + \frac{1}{\tilde{r}_i}$ . Thus from the embedding  $H^s \hookrightarrow L^\infty$  we have

$$\begin{aligned} & \|f(x, \Phi) - f(x, \Psi)\|_{L^{\tilde{l}_1} + L^{\tilde{l}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (\|\Phi\|_{L^\infty}^\ell + \|\Psi\|_{L^\infty}^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (2K^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}}. \end{aligned}$$

If  $n = 1$  and  $s = \frac{1}{2}$ , then for any  $1 < q_1, q_2 \leq \infty$ , we can find  $\tilde{r}_i, \tilde{l}_i \in [2, \infty)$  and  $\tilde{p}_i \gg 1$  such that  $\frac{1}{\tilde{l}_i} = \frac{1}{q_i} + \frac{\ell}{\tilde{p}_i} + \frac{1}{\tilde{r}_i}$ . So, we have

$$\begin{aligned} & \|f(x, \Phi) - f(x, \Psi)\|_{L^{\tilde{l}_1} + L^{\tilde{l}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (\|\Phi\|_{L^{\tilde{p}_1} + L^{\tilde{p}_2}}^\ell + \|\Psi\|_{L^{\tilde{p}_1} + L^{\tilde{p}_2}}^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}} \\ & \leq C\|b\|_{L^{q_1} + L^{q_2}} (2K^\ell) \|\Phi - \Psi\|_{L^{\tilde{r}_1} + L^{\tilde{r}_2}}. \end{aligned}$$

This proves Proposition 4.1. □

#### 4.1.1. 1-d uniqueness.

**Proposition 4.2.** *Let  $n = 1$ ,  $\frac{1}{2} < s < 1$ , and  $0 < \ell < \infty$ . Suppose that  $V$  and  $f$  satisfy the conditions of Proposition 4.1 and further  $V, b \in L^\infty$ . Then the  $H^s$ -weak solution to (1.1) is unique.*

*Proof of Proposition 4.2.* The solution  $\Phi$  constructed in Proposition 4.1 satisfies the integral equation

$$(4.1) \quad \Phi(t) = U(t)\varphi - i \int_0^t U(t-t')N(\cdot, \Phi(t')) dt' \quad \text{a.e. } t \in (-T_{\min}, T_{\max}),$$

where  $U(t) = e^{it(-\Delta)^s}$ . Let  $\Psi$  be another weak solution of (1.1) with the same initial data as  $\Phi$  on the interval  $[-T_1, T_2] \subset (-T_{\min}, T_{\max})$ . Assume that  $\|\Phi\|_{L^\infty(-T_1, T_2; H^s)} + \|\Psi\|_{L^\infty(-T_1, T_2; H^s)} \leq K$ . Then for any interval  $[-t_1, t_2] \subset [T_1, T_2]$  we have

$$\begin{aligned} & \|\Phi - \Psi\|_{L^\infty(-t_1, t_2; L^2)} \\ & \leq C \int_{-t_1}^{t_2} (\|V(\Phi - \Psi)\|_{L^2} + \|b(|\Phi|^\ell + |\Psi|^\ell)|\Phi - \Psi\|_{L^2}) dt' \\ & \leq C \int_{-t_1}^{t_2} (\|V\|_{L^\infty} + \|b\|_{L^\infty}(\|\Phi\|_{L^\infty}^\ell + \|\Psi\|_{L^\infty}^\ell)) \|\Phi - \Psi\|_{L^2} dt' \\ & \leq C(t_1 + t_2)(\|V\|_{L^\infty} + \|b\|_{L^\infty}(2K^\ell)) \|\Phi - \Psi\|_{L^\infty(-t_1, t_2; L^2)}. \end{aligned}$$

So  $\Phi = \Psi$  on  $[-t_1, t_2]$  for sufficiently small  $t_1, t_2$ . Let  $I = (-a, b)$  be the maximal interval of  $[-T_1, T_2]$  with  $\|\Phi - \Psi\|_{L^\infty(-c, d; L^2)} = 0$  for  $c < a, d < b$ . Suppose that  $a < T_1$  or  $b < T_2$ . Without loss of generality, we may assume that  $a < T_1$  and  $b < T_2$ . Then for a small  $\varepsilon > 0$  we can find  $a < t_1 < T_1, b < t_2 < T_2$  such that

$$\begin{aligned} & \|\Phi - \Psi\|_{L^\infty(-t_1, t_2; L^2)} \\ & \leq C \int_{-t_1}^{t_2} (\|V\|_{L^\infty} + \|b\|_{L^\infty}(\|\Phi\|_{L^\infty}^\ell + \|\Psi\|_{L^\infty}^\ell)) \|\Phi - \Psi\|_{L^2} dt' \\ & \leq C(\|V\|_{L^\infty} + \|b\|_{L^\infty}(2K^\ell)) \int_{-t_1, t_2} \|\Phi - \Psi\|_{L^2} dt' \\ & = C(\|V\|_{L^\infty} + \|b\|_{L^\infty}(2K^\ell))(t_1 + t_2 - a - b) \|\Phi - \Psi\|_{L^\infty(-t_1, t_2; L^2)} \\ & \leq (1 - \varepsilon) \|\Phi - \Psi\|_{L^\infty(-t_1, t_2; L^2)}. \end{aligned}$$

This contradicts the maximality of  $I$ . Thus  $I = [-T_1, T_2]$ . Since  $[-T_1, T_2]$  is arbitrarily taken in  $(-T_{\min}, T_{\max})$ , we finally get the whole uniqueness.  $\square$

4.1.2. *Conditional uniqueness for  $n \geq 2$ .* The weak solution can be shown to be unique under an weighted integrability condition. For this purpose we introduce a mixed norm  $\|h\|_{L_p^m L_{\sigma}^{\tilde{m}}}$ , defined by  $(\int_0^\infty (\int_{S^1} |h(\rho\sigma)|^{\tilde{m}} d\sigma)^{\frac{m}{\tilde{m}}} \rho^{n-1} d\rho)^{\frac{1}{m}}$  for  $1 \leq m, \tilde{m} < \infty$ . The case  $m = \infty$  or  $\tilde{m} = \infty$  can be defined in the usual way. We set  $\ell_0 = \frac{2s-1}{2s(1-s)}$  if  $n = 2$  and  $\frac{n(2s-1)}{(n-2s)(n-1)}$  if  $n \geq 3$ , and set  $\delta = \frac{n+2s}{q} - \frac{n}{2}$ . Then we have the following.

**Proposition 4.3.** *Let  $n \geq 2$ ,  $\frac{1}{2} < s < 1$ , and  $0 < \ell < \ell_0$  if  $n = 2$ . Suppose that  $V$  and  $f$  satisfy the conditions of Proposition 4.1 and further that  $|x|^\delta V \in L_\rho^{m_1} L_\sigma^{\tilde{m}_1}$ ,  $|x|^\delta b \in L_\rho^{m_2} L_\sigma^{\tilde{m}_2}$  for*

$$\frac{1}{m_1} = \frac{1}{2} - \frac{1}{q}, \quad \frac{1}{\tilde{m}_1} = \frac{\frac{2s}{q} - \frac{1}{2}}{n-1}$$

and

$$\frac{1}{m_2} = \frac{1}{2} - \frac{1}{q} - \frac{\ell(n-2s)}{2n}, \quad \frac{1}{\tilde{m}_2} = \frac{\frac{2s}{q} - \frac{1}{2}}{n-1} - \frac{\ell(n-2s)}{2n}$$

with  $\frac{1}{2} + \frac{(n-1)\ell(n-2s)}{2n} \leq \frac{2s}{q} \leq s$ . Then the  $H^s$ -weak solution to (1.1) is unique.

*Proof of Proposition 4.3.* For the uniqueness we will use the following weighted Strichartz estimate (see for instance Lemma 6.2 of [3] and Lemma 2 of [4]).

**Lemma 4.4.** *Let  $n \geq 2$  and  $2 \leq q < 4s$ . Then we have*

$$(4.2) \quad \| |x|^{-\delta} U(\cdot) \varphi \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \leq C \|\varphi\|_{L^2},$$

where  $\delta = \frac{n+2s}{q} - \frac{n}{2}$ ,  $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{1}{n-1} \left( \frac{2s}{q} - \frac{1}{2} \right)$  and  $C$  is independent of  $t_1, t_2$ .

In [3] it was shown that

$$\| |x|^{-\delta} D_\sigma^{\frac{2s}{q} - \frac{1}{2}} U(\cdot) \varphi \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^2)} \leq C \|\varphi\|_{L^2}.$$

The inequality (4.2) can be derived by Sobolev embedding on the unit sphere. Here  $D_\sigma = \sqrt{1 - \Delta_\sigma}$ ,  $\Delta_\sigma$  is the Laplace-Beltrami operator on the unit sphere.

We first consider the 2-d case. From (4.2) one can readily deduce that

$$(4.3) \quad \| |x|^{-\delta} \int_0^t U(t-t') g(t') dt' \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \lesssim \|g\|_{L^1(-t_1, t_2; L^2)}$$

Set  $g = N(x, \Phi) - N(x, \Psi)$ . Then from (4.1) we have

$$\begin{aligned} & \| |x|^{-\delta} (\Phi - \Psi) \|_{L_{(-t_1, t_2)}^q L_\rho^q L_\sigma^{\tilde{q}}} \\ & \lesssim \int_{-t_1}^{t_2} \|N(\cdot, \Phi) - N(\cdot, \Psi)\|_{L^2} dt' \\ & \lesssim \int_{-t_1}^{t_2} \|V(\Phi - \Psi)\|_{L^2} dt' + \int_{-t_1}^{t_2} \|b(\cdot)(|\Phi|^\ell + |\Psi|^\ell)|\Phi - \Psi\|_{L^2} dt'. \end{aligned}$$

By Hölder's inequality with  $\frac{1}{2} = \frac{1}{q} + \frac{1}{m_1} = \frac{1}{q} + \frac{1}{\tilde{m}_1}$  and  $\frac{1}{2} = \frac{1}{m_2} + \frac{\ell(n-2s)}{2n} + \frac{1}{q} = \frac{1}{\tilde{m}_2} + \frac{\ell(n-2s)}{2n} + \frac{1}{q}$  we have

$$\begin{aligned} & \| |x|^{-\delta} (\Phi - \Psi) \|_{L_{(-t_1, t_2)}^q L_\rho^q L_\sigma^{\tilde{q}}} \\ & \lesssim \| |x|^\delta V \|_{L_\rho^{m_1} L_\sigma^{\tilde{m}_1}} \int_{-t_1}^{t_2} \| |x|^{-\delta_1} (\Phi - \Psi) \|_{L_\rho^q L_\sigma^{\tilde{q}}} dt' \\ & \quad + \int_{-t_1}^{t_2} \| |\cdot|^\delta b \|_{L_\rho^{m_2} L_\sigma^{\tilde{m}_2}} (\|\Phi\|_{L^{\frac{2}{1-2s}}}^\ell + \|\Psi\|_{L^{\frac{2}{1-2s}}}^\ell) \| |x|^{-\delta} (\Phi - \Psi) \|_{L_\rho^q L_\sigma^{\tilde{q}}} dt'. \end{aligned}$$

Our  $q$  and  $\ell$  for  $n = 2$  guarantee the well-definedness of the Hölder exponents  $m_1, \tilde{m}_1, m_2, \tilde{m}_2$ .

If  $(-t_1, t_2) \subset [-T_1, T_2]$  and  $\|\Phi\|_{L^\infty(-T_1, T_2; H^s)} + \|\Psi\|_{L^\infty(-T_1, T_2; H^s)} \leq K$ , then by Sobolev's and Hölder's inequalities we have

$$\||x|^{-\delta}(\Phi - \Psi)\|_{L^q_{(-t_1, t_2)} L^q_\rho L^{\tilde{q}}_\sigma} \leq C_{V, f, K}(t_1 + t_2)^{1-\frac{1}{\tilde{q}}} \||x|^{-\delta}(\Phi - \Psi)\|_{L^q_{(-t_1, t_2)} L^q_\rho L^{\tilde{q}}_\sigma},$$

where  $C_{V, f, K} = C \left( \|\cdot\|^\delta V\|_{L^{m_1}_\rho L^{\tilde{m}_1}_\sigma} + \|\cdot\|^\delta b\|_{L^{m_2}_\rho L^{\tilde{m}_2}_\sigma} (2K^\ell) \right)$ . So  $\Phi = \Psi$  on  $[-t_1, t_2]$  for sufficiently small  $t_1, t_2$ . By the same argument as in 1-d case we can extend this uniqueness to  $(-T_{min}, T_{max})$ .

We can proceed with the almost same way as the proof of uniqueness for high-d case. The only difference is the range of  $\ell$ . For the proof we need  $\frac{1}{m_2}, \frac{1}{\tilde{m}_2} \geq 0$ , for which we must have

$$\ell \leq \frac{n(2s-1)}{(n-1)(n-2s)}, \quad \ell < \frac{n(2s-1)}{2s(n-2s)}, \quad n \geq 2,$$

respectively. In 2-d, the former is bigger than the latter and vice versa in high-d.  $\square$

**4.1.3. Unconditional uniqueness.** If we restrict  $q$  in Proposition 4.3, then we can get the unconditional uniqueness as follows.

**Corollary 4.5.** *If  $n = 2$ ,  $\frac{1}{4s} < \frac{1}{q} \leq \min\left(\frac{1}{2} - \frac{\ell(1-s)}{s}, \frac{1}{1+2s}\right)$ , then the uniqueness as in Proposition 4.3 occurs in  $C((-T_{min}, T_{max}); H^s)$ .*

*Proof of Corollary 4.5.* In view of the proof of Proposition 4.3, we have only to show that

$$C((-T_{min}, T_{max}); H^s) \subset L^q_{loc}(-T_{min}, T_{max}; |x|^{\frac{2+2s}{q}-1} L^q_\rho L^{\tilde{q}}_\sigma),$$

where  $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{1}{n-1} \left( \frac{2s}{q} - \frac{1}{2} \right)$ . Since  $q \geq 1 + 2s$ ,  $\tilde{q} \leq q$  and thus

$$L^q_{loc}(-T_{min}, T_{max}; |x|^{\frac{2+2s}{q}-1} L^q_x) \subset L^q_{loc}(-T_{min}, T_{max}; |x|^{\frac{2+2s}{q}-1} L^q_\rho L^{\tilde{q}}_\sigma).$$

Using Hardy inequality<sup>2</sup> that  $\||x|^{-\alpha} u\|_{L^p} \lesssim \|u\|_{\dot{H}^{\alpha+n(\frac{1}{2}-\frac{1}{p})}}$  for  $0 < \alpha < \frac{n}{p}$  and  $2 \leq p < \infty$ , since  $1 - \frac{2+2s}{q} < 0$  and  $q \geq 2 > 2s$  for the above  $q$ , we get

$$\||x|^{1-\frac{2+2s}{q}} u\|_{L^q_x} \lesssim \|u\|_{\dot{H}^{\frac{2s}{q}}} \lesssim \|u\|_{H^s}.$$

By this we deduce that  $H^s \subset |x|^{\frac{2+2s}{q}-1} L^q_x$ . This completes the proof of corollary.  $\square$

By exactly the same way, we have the following.

**Corollary 4.6.** *If  $n \geq 3$ ,  $\max\left(\frac{1}{2}, \frac{ns}{n+2s}, \frac{1}{2} + \frac{(n-1)\ell(n-2s)}{2n}\right) < \frac{2s}{q} \leq \frac{ns}{n+2s-1}$  and  $0 < \ell \leq \frac{n(2s-1)}{(n-2s)(n+2s-1)}$ , then the uniqueness as in Proposition 4.3 occurs in  $C((-T_{min}, T_{max}); H^s)$ .*

<sup>2</sup>Such inequality can be shown by the interpolation between the estimates  $\||x|^{-\alpha} u\|_{L^2} \lesssim \|u\|_{\dot{H}^\alpha}$  and  $\|u\|_{BMO} \lesssim \|u\|_{\dot{H}^{\frac{n}{2}}}$ .

**4.2. Well-posedness.** By using the argument of [2] one can show that the uniqueness implies actually well-posedness and conservation laws:

- $\Phi \in C(-T_{min}, T_{max}; H^s) \cap C^1(-T_{min}, T_{max}; H^{-s}),$
- $\Phi$  depends continuously on  $\varphi$  in  $H^s,$
- $M(\Phi(t)) = M(\varphi)$  and  $J(\Phi(t)) = J(\varphi) \quad \forall t \in (-T_{min}, T_{max}).$

We leave the details to the readers. In this section we remark on the global well-posedness.

**4.2.1. Remarks on global well-posedness.** We discussed that the conditions on  $V$  and  $f$  of Propositions 4.1-4.3 give the uniqueness and local well-posedness of (1.1). In this section we study some conditions guaranteeing the global well-posedness.

- (1) If  $f(x, |\tau|) \leq 0$  for all  $x$  and  $\tau$ , and  $\Phi$  is the unique solution to (1.1), then

$$J(\varphi) = J(\Phi) \geq \frac{1}{2} \|(\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2 - \|V\|_{L^{p_1} + L^{p_2}} \|\Phi\|_{L^{2p'_1} + L^{2p'_2}}.$$

Since  $2 < 2p'_i < s^*$ , from Gagliardo-Nirenberg inequality it follows that for some  $0 < \theta_i < 1$

$$\begin{aligned} J(\varphi) &\geq \frac{1}{2} \|(\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2 \\ &- C \|V\|_{L^{p_1} + L^{p_2}} (\|\varphi\|_{L^2}^{2(1-\theta_1)} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^{2\theta_1} + \|\varphi\|_{L^2}^{2(1-\theta_2)} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^{2\theta_2}). \end{aligned}$$

By Young's inequality we get  $\|\Phi(t)\|_{H^s}^2 \leq C_V \|\varphi\|_{L^2}^2 + 4J(\varphi)$  for all  $t \in (-T_{min}, T_{max})$ .

The continuity argument implies the global well-posedness that  $T_{min} = T_{max} = \infty$ .

- (2) If  $\ell < \frac{4s}{n}$  and  $q_i > \frac{2n}{4s-n\ell} (> \frac{2n}{2n-(\ell+2)(n-2s)})$ , then Gagliardo-Nirenberg inequality gives the uniform bound of  $\|\Phi(t)\|_{H^s}$ . More precisely,

$$\begin{aligned} J(\varphi) = J(\Phi) &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2 - \|V\|_{L^{p_1} + L^{p_2}} \|\Phi\|_{L^{2p'_1} + L^{2p'_2}}^2 \\ &- \|a\|_{L^{q_1} + L^{q_2}} \|\Phi\|_{L^{(\ell+2)q'_1} + L^{(\ell+2)q'_2}}^{\ell+2}. \end{aligned}$$

Since  $q_i > \frac{2n}{4s-n\ell}$ ,  $2 < (\ell+2)q'_i < \frac{2n}{n-2s}$ . As above we get  $\|\Phi(t)\|_{H^s}^2 \leq C_{V,a} \|\varphi\|_{L^2}^2 + 4J(\varphi)$  for all  $t \in (-T_{min}, T_{max})$  and thus global well-posedness.

- (3) If  $n = 1$ ,  $\frac{1}{2} < s < 1$ ,  $V = 0$  and  $f(x, \Phi) = \lambda |\Phi|^2 \Phi$ , then in [10] the authors showed global well-posedness in  $L^2$  by using Bourgain space argument. For another global well-posedness we refer the reader to [13], where a problem with  $s = \frac{1}{4}$  and combined cubic nonlinearity is treated.

- (4) If  $\ell = \frac{4s}{n}$ , then since

$$\begin{aligned} J(\varphi) = J(\Phi) &\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2 - \|V\|_{L^{p_1} + L^{p_2}} \|\Phi\|_{L^{2p'_1} + L^{2p'_2}}^2 \\ &- \|a\|_{L^\infty} \|\varphi\|_{L^2}^{\frac{4s}{n}} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^2}^2, \end{aligned}$$

we have  $\|\Phi(t)\|_{H^s}^2 \leq C \|\varphi\|_{L^2}^2 + 4J(\varphi)$  for  $\|a\|_{L^\infty} \|\varphi\|_{L^2}^{\frac{4s}{n}} < \frac{1}{8}$ .

- (5) If  $q$  and  $\ell$  satisfies the condition of Corollary 4.5 and 4.6, then the well-posedness is unconditional.

4.2.2. *Well-posedness of radial solutions.* From now on we consider the well-posedness of radial solutions to (1.1) when  $\ell \leq \frac{4s}{n-2s}$ . In [3] the authors considered the well-posedness for Hartree type nonlinearity by using various Strichartz estimates. Indeed, they utilized weighted or angularly regular Strichartz estimate to control the Hartree type nonlinearity. However, if the power type nonlinearity  $f$  is involved, then the situation is quite different. It is not easy to handle angular regularity for which we need a high regularity of  $f$ . To avoid this we assume the radial symmetry of  $f$  and initial data.

Let us introduce radial Strichartz estimate of  $U(t)$  (see [5]): for  $\frac{n}{2n-1} \leq s < 1$ ,  $2 \leq q \leq \infty$ ,  $2 \leq r < \infty$  with  $\frac{2s}{q} + \frac{n}{r} = \frac{n}{2}$  and  $(q, r) \neq (2, \frac{4n-2}{2n-3})$

$$(4.4) \quad \|U(\cdot)\varphi\|_{L^q(-T_1, T_2; L^r_x)} \lesssim \|\varphi\|_{L^2}.$$

We call such pair  $(q, r)$   $s$ -admissible one. The constant involved in (4.4) is independent of  $T_1, T_2$ . The estimate (4.4) can be extended to Besov type as follows:

$$(4.5) \quad \|U(\cdot)\varphi\|_{L^q(-T_1, T_2; B^s_r)} \lesssim \|\varphi\|_{H^s}.$$

Here  $B^s_r = B^s_{r,2}$  is the inhomogeneous Besov space. Using Christ-Kiselev lemma we get the inhomogeneous Strichartz estimates: Let  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  be  $s$ -admissible pairs with  $q > \tilde{q}$ . Then

$$(4.6) \quad \left\| \int_0^t U(t-t')g(t') dt' \right\|_{L^q(-T_1, T_2; B^s_r)} \lesssim \|g\|_{L^{\tilde{q}}_T B^s_{\tilde{r}}}.$$

Under the fractional and power type setting, an alternative Besov norm is useful, which is stated as follows: for  $0 < s < 1$ ,  $1 \leq r < \infty$

$$(4.7) \quad \|g\|_{B^s_r} \sim \|g\|_{L^r} + \left( \int_0^\infty (\alpha^{-s} \sup_{|y| \leq \alpha} \|g(\cdot + y) - g(\cdot)\|_{L^r})^2 \frac{d\alpha}{\alpha} \right)^{\frac{1}{2}}.$$

The following is the local well-posedness result.

**Proposition 4.7.** *Suppose that  $V(x) = V(|x|) \in L^\infty$ ,  $(-\Delta)^{\frac{s}{2}}V \in L^{\frac{n}{s}}$ ,  $f(x, z) = f(|x|, z)$ ,  $a, b \in L^\infty$  and  $\varphi(x) = \varphi(|x|) \in H^s$ . Let*

$$r_0 = \frac{n(\ell + 2)}{n + s\ell}, \quad q_0 = \frac{4s(\ell + 2)}{\ell(n - 2s)}$$

for  $\frac{n}{2n-1} \leq s < 1$ , and  $0 < \ell \leq \frac{4s}{n-2s}$ . Then there exists  $T_1, T_2 > 0$  such that (1.1) has a unique radial solution  $u \in C([-T_1, T_2]; H^s) \cap L^{q_0}(-T_1, T_2; B^s_{r_0})$ .

The pair  $(q_0, r_0)$  is  $s$ -admissible one and  $q_0 > 2$ .

*Proof of Proposition 4.7.* For simplicity we only consider the well-posedness on  $[0, T]$ . Let  $(X_T^\rho, d_X)$  be a metric space with metric  $d_X$  defined by

$$X_T^\rho = \{\Phi \in L_T^\infty H^s \cap L_T^{q_0} B^s_{r_0} : \Phi \text{ is radial and } \|\Phi\|_{L_T^\infty H^s \cap L_T^{q_0} B^s_{r_0}} \leq \rho\},$$

$$d_X(\Phi, \Psi) = \|\Phi - \Psi\|_{L_T^\infty L^2 \cap L_T^{q_0} L^{r_0}}.$$

$L_T^q \mathcal{B}$  denotes  $L_t^q([0, T]; \mathcal{B})$  for some positive  $T$  and Banach space  $\mathcal{B}$ . Since  $H^s$  and  $B_{r_0}^s$  are reflexive Banach space, one can readily show that  $X_T^\rho$  is complete. We define a mapping  $\mathcal{N}$  on  $X_T^\rho$  by

$$(4.8) \quad \mathcal{N}(\Phi)(t) = U(t)\varphi - i \int_0^t U(t-t')[N(\cdot, \Phi)](t') dt'.$$

We use the standard contraction mapping argument. For any  $\Phi \in X_T^\rho$  we have from (4.5) and (4.6) with  $q_0, r_0$  that

$$(4.9) \quad \|\mathcal{N}(\Phi)\|_{L_T^\infty H^s \cap L_T^{q_0} B_{r_0}^s} \lesssim \|U(\cdot)\varphi\|_{L_T^{q_0} B_{r_0}^s} + \|V\Phi\|_{L_T^1 H^s} + \|f(\cdot, \Phi)\|_{L_T^{q'_0} B_{r_0}^s}.$$

From the fractional Leibniz rule, we have

$$(4.10) \quad \begin{aligned} \|V\Phi\|_{L_T^1 H^s} &\lesssim T(\|V\|_{L^\infty} \|\Phi\|_{L_T^\infty H^s} + \|(-\Delta)^{\frac{s}{2}} V\|_{L^{\frac{n}{s}}} \|\Phi\|_{L_T^\infty L^{\frac{2n}{n-2s}}}) \\ &\lesssim T(\|V\|_{L^\infty} + \|(-\Delta)^{\frac{s}{2}} V\|_{L^{\frac{n}{s}}}) \|\Phi\|_{L_T^\infty H^s}. \end{aligned}$$

On the other hand, since  $\frac{1}{r'_0} = \frac{\ell}{r} + \frac{1}{r_0}$  for  $\frac{1}{r} = \frac{n-2s}{n(\ell+2)}$  which equals  $\frac{1}{r_0} - \frac{s}{n}$ , from the condition (1.2) and Sobolev embedding  $B_{r_0}^s \hookrightarrow L^r$  it follows that

$$\|f(\cdot, \Phi)\|_{L_x^{r'_0}} \lesssim \|\Phi\|_{L_x^r}^\ell \|\Phi\|_{L_x^{r_0}} \lesssim \|\Phi\|_{B_{r_0}^s}^{\ell+1}.$$

From (1.3) we have

$$\begin{aligned} &|f(x+y, \Phi(x+y)) - f(x, \Phi(x))| \\ &\lesssim \min(1, |y|) |\Phi(x+y)|^{\ell+1} + (|\Phi(x+y)|^\ell + |\Phi(x)|^\ell) |\Phi(x+y) - \Phi(x)|. \end{aligned}$$

So, we get as above

$$\begin{aligned} &\|f(\cdot + y, \Phi(\cdot + y)) - f(\cdot, \Phi(\cdot))\|_{L^{r'_0}} \\ &\lesssim \min(1, |y|) \|\Phi\|_{B_{r_0}^s}^{\ell+1} + \|\Phi\|_{B_{r_0}^s}^\ell \|\Phi(\cdot + y) - \Phi(\cdot)\|_{L^{r_0}}. \end{aligned}$$

Thus we get from (4.7)

$$(4.11) \quad \|f(\cdot, \Phi)\|_{B_{r_0}^s} \lesssim \|\Phi\|_{B_{r_0}^s}^{\ell+1} (1 + \left( \int_0^\infty [\alpha^{-s} \min(1, \alpha)]^2 \frac{d\alpha}{\alpha} \right)^{\frac{1}{2}}) \lesssim \|\Phi\|_{B_{r_0}^s}^{\ell+1}.$$

Now let us turn to the nonlinear estimate (4.9). We take Hölder's inequality in  $t$ -variable with

$$\frac{1}{q'_0} = \frac{\ell+1}{q_0} + \frac{1}{q_1}.$$

From the condition of  $\ell$  we have  $1/q_1 \geq 0$ , and  $1/q_1 = 0$  when  $\ell = \frac{4s}{n-2s}$ . Thus we get

$$\begin{aligned} \|\mathcal{N}(\Phi)\|_{L_T^\infty H^s \cap L_T^{q_0} B_{r_0}^s} &\leq \|U(\cdot)\varphi\|_{L_T^{q_0} B_{r_0}^s} + CT \|\Phi\|_{L_T^\infty H^s} + CT^{\frac{1}{q_1}} \|\Phi\|_{L_T^{q_0} B_{r_0}^s}^{\ell+1} \\ &\lesssim \|U(\cdot)\varphi\|_{L_T^{q_0} B_{r_0}^s} + CT\rho + CT^{\frac{1}{q_1}} \rho^{\ell+1}. \end{aligned}$$

If  $\ell < \frac{4s}{n-2s}$ , then since  $\|U(\cdot)\varphi\|_{L_T^{q_0} B_{r_0}^s} \lesssim \|\varphi\|_{H^s}$ ,  $T$  can be chosen to be dependent only on  $C$  and  $\|\varphi\|_{H^s}$  to guarantee  $\mathcal{N}(\Phi) \in X_T^\rho$ . If  $\ell = \frac{4s}{n-2s}$ , then we first choose  $T, \rho$  such that

$C(T\rho + T^{\frac{1}{q_1}}\rho^{\ell+1}) \leq \rho/2$  and then choose smaller  $T$  such that  $\|U(\cdot)\varphi\|_{L_T^{q_0} B_{r_0}^s} \leq \frac{\rho}{2}$ , which means  $\mathcal{N}(\Phi) \in X_T^\rho$ .

Now we show that  $\mathcal{N}$  is a Lipschitz map for sufficiently small  $T$ . Let  $\Phi, \Psi \in X_T^\rho$ . Then from the same estimates as above we have

$$\begin{aligned} d_X(\mathcal{N}(\Phi), \mathcal{N}(\Psi)) &\leq C\|V(\Phi - \Psi)\|_{L_T^1 H^s} + C\|f(\cdot, \Phi) - f(\cdot, \Psi)\|_{L_T^{q_0'} B_{r_0}^{r_0'}} \\ &\leq CT(\|V\|_{L^\infty} + \|(-\Delta)^{\frac{s}{2}} V\|_{L_T^{\frac{n}{s}}})\|\Phi - \Psi\|_{L_T^\infty H^s} \\ &\quad + CT^{\frac{1}{q_1}}(\|\Phi\|_{L_T^{q_0} B_{r_0}^s}^\ell + \|\Psi\|_{L_T^{q_0} B_{r_0}^s}^\ell)\|\Phi - \Psi\|_{L_T^{q_0} L^{r_0}} \\ &\leq C(T + T^{\frac{1}{q_1}}(2\rho^\ell))d_X(\Phi - \Psi). \end{aligned}$$

Thus for smaller  $T$  and  $\rho$  the mapping  $\mathcal{N}$  is a contraction and there is a fixed point  $\Phi$  of  $\mathcal{N}$  satisfying (4.1). The uniqueness and time continuity follows easily from the equation (4.1) and Strichartz estimate. We omit the details.  $\square$

**Remark 1.** *The mass and energy conservations are straightforward from the uniqueness. One can also show the conservation laws by the argument for Strichartz solutions of [15]. The global well-posedness follows easily from the conservations in case that  $f(x, |\tau|) \leq 0$  and  $\ell < \frac{4s}{n-2s}$ , or  $\ell < \frac{4s}{n}$ , or  $f(x, |\tau|) \leq 0$  and  $\ell = \frac{4s}{n-2s}$  and  $\|\varphi\|_{H^s}$  is small.*

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